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Global dimension of the endomorphism ring and $*^n$-modules

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Abstract

We show that, if T is a selfsmall and selforthogonal module over a noetherian ring R of finite global dimension with the endomorphism ring A, then $\operatorname{fd} T_A \leqslant \operatorname{gd} A \leqslant \operatorname{id}_R T + \operatorname{fd} T_A$. Applying the result we give answers to two questions left in [J. Wei et al., J. Algebra 168 (2) (2003) 404–418] concerning basic properties of $*^n$-modules, by showing that the flat dimension of a $*^n$-module with $n \geqslant 3$ over its endomorphism ring can even be arbitrarily far from the integer n while the flat dimension of a $*^2$-module over its endomorphism ring is always bounded by the integer 2 and showing that $*^n$-modules are not finitely generated in general, even in case $n = 2$.

Keywords: Global dimension; Selforthogonal module; $*^n$-Module

1. Introduction and preliminaries

The tilting theory plays an important role in the representation of Artin algebra. Let R be an Artin algebra and T a tilting R-module with $A = \operatorname{End}_R T$ (by a tilting module we mean the tilting module in sense of [9] throughout the paper). The purpose of the tilting theory is to compare R-mod (the category of finitely generated R-modules) with A-mod. One aspect of these is on the estimate of the global dimension of the endomorphism algebra A.

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A well-known result is that $\mathrm{gd}\, R - \mathrm{pd}\,_R T \leqslant \mathrm{gd}\, A \leqslant \mathrm{gd}\, R + \mathrm{pd}\,_R T$, see for instance [8,9]. The result was improved in [7] where it was shown that $\mathrm{id}\,_R T \leqslant \mathrm{gd}\, A \leqslant \mathrm{pd}\,_R T + \mathrm{id}\,_R T$. In the first part of this paper, we will study some more general case in the sense that we only assume that T is selfsmall and selforthogonal and that T is over any associated ring with identity, not only Artin algebras. A special case of our results shows that, if T is a selfsmall and selforthogonal module over a noetherian ring R of finite global dimension with the endomorphism ring A, then $\mathrm{fd}\, T_A \leqslant \mathrm{gd}\, A \leqslant \mathrm{id}\,_R T + \mathrm{fd}\, T_A$ (Corollary 2.6). Note that, for a tilting module T over an Artin algebra R with $A = \mathrm{End}\,_R T$, it always holds that $\mathrm{pd}\,_R T = \mathrm{pd}\, T_A = \mathrm{fd}\, T_A$, so our result extends the upper-bound-part of the corresponding one in [7].

The important tool for our investigation is the theory of $*^n$-modules. Recall that an R-module T is a $*^n$-module provided that T is selfsmall, $(n+1)$-quasi-projective and that $\mathrm{Pres}^n(T) = \mathrm{Pres}^{n+1}(T)$ [16]. The notion of $*^n$-modules is a natural generalization of both $*$-modules (see for instance [3,4,10] etc.) and tilting modules of projective dimension $\leqslant n$. In fact, $*$-modules are just $*^1$-modules while tilting modules of projective dimension $\leqslant n$ are just $*^n$-modules which admit finitely generated projective resolutions and n-present all injective modules [2,16].

In [16], some questions on basic properties of $*^n$-modules were left. The following are two questions among them.

Question 1. *Is the flat dimension of a $*^n$-module over its endomorphism ring is always bounded by n?*

Question 2. *Are all $*^n$-modules finitely generated?*

We recall that all $*$-modules are finitely generated [13] and that the flat dimension of a $*$-module over its endomorphism ring is always bounded by 1 [14]. Note that it was also proved that, for a $*^n$-module T with the endomorphism ring A, it always holds that $\mathrm{Ker}\,\mathrm{Tor}^A_{i\geqslant 1}(T,-) = \mathrm{Ker}\,\mathrm{Tor}^A_{1\leqslant i\leqslant n}(T,-)$ [15].

Our investigation of the estimate of global dimensions of endomorphism rings of self-orthogonal modules turns out to be very useful to answer questions mentioned above. Indeed, it is shown, in the second part of this paper, that the flat dimension of a $*^n$-module with $n \geqslant 3$ over its endomorphism ring can even be arbitrarily far from the integer n (Proposition 3.3). However, the flat dimension of a $*^2$-module over its endomorphism ring is always bounded by the integer 2 (Proposition 3.5). In particular, we obtain that, if T is a w-Σ-quasi-projective $*^2$-module over the ring R and Q is any injective cogenerator of R-Mod, then the "dual" module $T^* = \mathrm{Hom}_R(T, Q)$ is a cotilting module over the endomorphism ring of T (Corollary 3.6). Finally, we give an example to show that a $*^n$-module is not finitely generated in general, even in case $n = 2$ (Example 3.10). In fact, the rational $\mathbf{Q}$, as a $\mathbf{Z}$-module ($\mathbf{Z}$ denotes the ring of all integers), is just such a module.

Throughout this paper, all rings will be associated with non-zero identity and modules will be left modules without explicit mentions. For a ring R, R-Mod (Mod-R) denotes the category of all left (right) R-modules. By a subcategory, we mean a full subcategory closed under isomorphisms.

An R-module T is selfsmall if the canonical morphism

$$\mathrm{Hom}_R(T,T)^{(X)} \to \mathrm{Hom}_R\big(T,T^{(X)}\big)$$

is an isomorphism.

From now on, we fix T a selfsmall R-module with the endomorphism ring A and denote that $H_T = \mathrm{Hom}_R(T,-)$. Note that T is also a right A-module. Let Q be any injective cogenerator of R-Mod. We fix $T^* = \mathrm{Hom}_R(T,Q)$. Then T^* is an A-module.

We denote that

$$T^{\perp} = \big\{M \in R\text{-Mod} \mid \mathrm{Ext}^i_R(T,M) = 0 \text{ for all } i \geqslant 1\big\} \quad \text{and}$$
$$\mathrm{Ker}\,\mathrm{Tor}^A_{i\geqslant 1}(T,-) = \big\{M \in A\text{-Mod} \mid \mathrm{Tor}^A_i(T,M) = 0 \text{ for all } i \geqslant 1\big\}.$$

For a fixed integer n, the subcategory $\mathrm{Ker}\,\mathrm{Tor}^A_{1\leqslant i\leqslant n}(T,-)$ is similarly defined. Also we denote by $\mathrm{Ad}\,T$ the class of modules isomorphic to direct sums of copies of the R-module T and by $\mathrm{Add}\,T$ the class of modules isomorphic to direct summands of modules in $\mathrm{Ad}\,T$. Furthermore, we denote that

$$\widehat{\mathrm{Add}\,T} = \big\{M \in R\text{-}\mathrm{Mod} \mid \text{there exists an exact sequence } 0 \to T_m \to \cdots \to T_0 \to M \to 0$$
$$\text{for some } m, \text{ where } T_i \in \mathrm{Add}\,T \text{ for each } i\big\}.$$

We say that an R-module T is selforthogonal if $\mathrm{Ext}^i_R(T,T') = 0$ for all $i \geqslant 1$ and all $T' \in \mathrm{Add}\,T$. If T has a finitely generated projective resolution, then T is selforthogonal is equivalent to say that $\mathrm{Ext}^i_R(T,T) = 0$ for all $i \geqslant 1$ [16].

An R-module M is n-presented by T if there exists an exact sequence $T_n \to \cdots \to T_2 \to T_1 \to M \to 0$ with $T_i \in \mathrm{Add}\,T$ for each i. We denote by $\mathrm{Pres}^n(T)$ the category of all R-modules n-presented by T. Of course, for every n, we have that $\mathrm{Pres}^{n+1}(T) \subseteq \mathrm{Pres}^n(T)$. Note that $\mathrm{Pres}^2(T)$ and $\mathrm{Pres}^1(T)$ are just familiar subcategories $\mathrm{Pres}(T)$ and $\mathrm{Gen}(T)$ respectively. We denote that $\mathrm{Copres}(T^*) = \{N \in A\text{-Mod} \mid$ there exists an exact sequence $0 \to N \to K_1 \to K_2$ with K_1, K_2 products of copies of $T^*\}$ and that $\mathrm{Cogen}(T^*) = \{N \in A\text{-Mod} \mid N$ can be embedded in a product of copies of $T^*\}$.

T is said to be (n,t)-quasi-projective (here we assume that $n \geqslant t \geqslant 1$) if, for any exact sequence $0 \to M \to T_t \to \cdots \to T_1 \to N \to 0$ with $M \in \mathrm{Pres}^{n-t}(T)$ and $T_i \in \mathrm{Add}\,T$ for each i, the induced sequence $0 \to H_T M \to H_T T_t \to \cdots \to H_T T_1 \to H_T N \to 0$ is exact [15]. It is easy to see that, if T is (n,t)-quasi-projective, then T is also (m,s)-quasi-projective, for all m,s such that $m \geqslant n$ and $1 \leqslant s \leqslant t+m-n$. Note that notions of $(1,1)$-quasi-projective, $(2,1)$-quasi-projective, $(2,2)$-quasi-projective and $(n,1)$-quasi-projective respectively are just notions of Σ-quasi-projective [6] [11], w-Σ-quasi-projective [3], semi-Σ-quasi-projective [11] and n-quasi-projective [16], respectively.

It is well known that $(T \otimes_A -, H_T)$ is a pair of adjoint functors and there are the following canonical homomorphisms for any R-module M and any A-module N:

$$\rho_M : T \otimes_A H_T M \to M \quad \text{by} \quad t \otimes f \to f(t),$$
$$\sigma_N : N \to H_T(T \otimes_A N) \quad \text{by} \quad n \to [t \to t \otimes n].$$

We denote by $\mathrm{Stat}(T)$ the class of R-modules M such that ρ_M is an isomorphism and by $\mathrm{Costat}(T)$ the class of A-modules N such that σ_N is an isomorphism.

Recall from [2] that an A-module K is 2-cotilting (1-cotilting, respectively) if

$$\begin{aligned}\mathrm{Copres}(K)\ \big(\mathrm{Cogen}(K),\ \text{respectively}\big) &= \mathrm{Ker}\,\mathrm{Ext}_A^{i\geqslant 1}(-,K)\\ &:= \big\{M \in A\text{-Mod} \mid \mathrm{Ext}_A^i(M,K)=0 \text{ for all } i \geqslant 1\big\}\end{aligned}$$

(the definition is not the original one, but is equivalent to it by [2]). This definition is used throughout the paper. Note that in case A is an Artin algebra, finitely generated 1-cotilting modules coincide with usual cotilting modules (of injective dimension $\leqslant 1$) in sense of [8].

Throughout the paper, $\mathrm{gd}\,R$ denotes the (left) global dimension of the ring R. We denote by $\mathrm{pd}\,_RT$ ($\mathrm{id}\,_RT$, $\mathrm{fd}\,T_A$, respectively) the projective (injective, flat, respectively) dimension of the module $_RT$ ($_RT$, T_A, respectively).

2. Global dimension of endomorphism rings

For any $M \in \widehat{\mathrm{Add}\,T}$, there is some m such that there is an exact sequence $0 \to T_m \to \cdots \to T_0 \to M \to 0$ with $T_i \in \mathrm{Add}\,T$ for each i, by the definition. We denote $T\text{-res.dim}(M)$ to be the minimal integer among such m. Then we have the following useful lemma.

Lemma 2.1. *Let T be a selforthogonal R-module. Then $T\text{-res.dim}(M) = \mathrm{pd}\,_AH_TM$, for any $M \in \widehat{\mathrm{Add}\,T}$.*

Proof. Assume that $m = T\text{-res.dim}(M)$, then there is an exact sequence of minimal length $0 \to T_m \to^{f_m} \cdots \to T_0 \to M \to 0$ with $T_i \in \mathrm{Add}\,T$ for each i. By applying the functor H_T to the sequence, we obtain an induced exact sequence $0 \to H_TT_m \to^{H_Tf_m} \cdots \to H_TT_0 \to H_TM \to 0$, since T is selforthogonal. The last sequence in fact is a projective resolution of the A-module H_TM. Hence $\mathrm{pd}\,_AH_TM \leqslant m = T\text{-res.dim}(M)$. If $\mathrm{pd}\,_AH_TM < m$, then it is easy to check that $\mathrm{Coker}\,H_Tf_m$ must be a projective A-module. Now after applying the functor $T \otimes_A -$ to the sequence $0 \to H_TT_m \to^{H_Tf_m} H_TT_{m-1} \to \mathrm{Coker}\,H_Tf_m \to 0$, we deduce that $\mathrm{Coker}\,f_m \simeq \mathrm{Coker}\,T \otimes_A H_Tf_m = T \otimes_A \mathrm{Coker}\,H_Tf_m \in \mathrm{Add}\,T$. This shows that $T\text{-res.dim}(M) < m$, a contradiction. In conclusions, we have that $T\text{-res.dim}(M) = \mathrm{pd}\,_AH_TM$, for any $M \in \widehat{\mathrm{Add}\,T}$. $\square$

Under some additional conditions, $T\text{-res.dim}(M)$ will be bounded by a fixed number for all $M \in \widehat{\mathrm{Add}\,T}$.

Lemma 2.2. *Let T be a selforthogonal R-module with $\mathrm{id}\,_RT' \leqslant s < \infty$ for all $T' \in \mathrm{Add}\,T$. Then $T\text{-res.dim}(M) \leqslant s$ for any $M \in \widehat{\mathrm{Add}\,T}$. In particular, $\mathrm{pd}\,_AH_TM \leqslant s$.*

Proof. For any $M \in \widehat{\mathrm{Add}\,T}$, there is an exact sequence $0 \to T_m \to^{f_m} \cdots \to T_0 \to^{f_0} M \to 0$ with $T_i \in \mathrm{Add}\,T$ for each i. If $m \leqslant s$, then we have nothing to say. So we assume that $m > s$. Let $K_i = \operatorname{Im} f_i$ for each $0 \leqslant i \leqslant m$. Note that $K_m = T_m$ and $K_0 = M$ in this case. Since T is selforthogonal, we have that

$$\operatorname{Ext}_R^j(T_i, K_m) = 0 \quad \text{for all } j \geqslant 1 \text{ and all } 0 \leqslant i \leqslant m.$$

Then, by applying the functor $\operatorname{Hom}_R(-, K_m)$ to the sequence, we obtain that

$$\operatorname{Ext}_R^1(K_{m-1}, K_m) \simeq \operatorname{Ext}_R^2(K_{m-2}, K_m) \simeq \cdots \simeq \operatorname{Ext}_R^m(K_0, K_m),$$

by dimension shifting. By assumptions, $\operatorname{id}_R T' \leqslant s$ for all $T' \in \operatorname{Add} T$, so we have that $\operatorname{id} K_m \leqslant s < m$. It follows that $\operatorname{Ext}_R^1(K_{m-1}, K_m) \simeq \operatorname{Ext}_R^m(K_0, K_m) = 0$. Hence the sequence $0 \to T_m \to^{f_m} T_{m-1} \to K_{m-1} \to 0$ splits. This shows that T-res.dim(M) must be not more than s. $\square$

Lemma 2.3. *Let T be a selforthogonal R-module with $\operatorname{pd}_R T \leqslant n$. Then $\operatorname{Pres}^n(T) \subseteq T^{\perp}$ and T is $(n+1)$-quasi-projective.*

Proof. Take any $M \in \operatorname{Pres}^n(T)$, then we have an exact sequence $T_n \to^{f_n} \cdots \to T_1 \to^{f_1} M \to 0$. Let $M_i = \operatorname{Ker} f_i$ for each $1 \leqslant i \leqslant n$. Since T is selforthogonal, we have that

$$\operatorname{Ext}_R^j(T, T_i) = 0 \quad \text{for all } j \geqslant 1 \text{ and all } 1 \leqslant i \leqslant n.$$

Then, by applying the functor H_T to the sequence, we obtain that

$$\operatorname{Ext}_R^j(T, M) \simeq \operatorname{Ext}_R^{j+1}(T, M_1) \simeq \cdots \simeq \operatorname{Ext}_R^{j+n}(T, M_n) \quad \text{for all } j \geqslant 1,$$

by dimension shifting. Since $\operatorname{pd}_R T \leqslant n$, we deduce that

$$\operatorname{Ext}_R^j(T, M) \simeq \operatorname{Ext}_R^{j+n}(T, M_n) = 0 \quad \text{for all } j \geqslant 1.$$

Hence $M \in T^{\perp}$, and consequently, $\operatorname{Pres}^n(T) \subseteq T^{\perp}$.

Now consider any exact sequence $0 \to K \to T_N \to N \to 0$ with $K \in \operatorname{Pres}^n(T)$ and $T_N \in \operatorname{Add} T$. By the previous proof, we see that $K \in T^{\perp}$. Hence, by applying the functor H_T to the last sequence, we obtain that the induced sequence $0 \to H_T K \to H_T T_N \to H_T N \to 0$ is exact, i.e., T is $(n+1)$-quasi-projective. $\square$

The following result shows that a selfsmall and selforthogonal module over a ring of finite global dimension is always a $*^m$-module for some integer m.

Proposition 2.4. *Let T be a selforthogonal R-module with $\operatorname{pd}_R T \leqslant n$ $(n \geqslant 1)$. If $\operatorname{gd} R = d < \infty$, then $\operatorname{Pres}^{n+d}(T) = \widehat{\operatorname{Add} T}$. In particular, T is a $*^{n+d}$-module.*

Proof. Obviously we have that $\widehat{\operatorname{Add} T} \subseteq \operatorname{Pres}^{n+d}(T)$. We now show that $\operatorname{Pres}^{n+d}(T) \subseteq \widehat{\operatorname{Add} T}$ too.

For any $M \in \mathrm{Pres}^{n+d}(T)$, we have an exact sequence $T_{n+d} \to^{f_{n+d}} \cdots \to T_d \to^{f_d} \cdots \to T_1 \to^{f_1} M \to 0$ with $T_i \in \mathrm{Add}\,T$ for each $1 \leqslant i \leqslant n+d$. Denote that $M_i = \mathrm{Ker}\, f_i$ for each i, then we see that $M_d \in \mathrm{Pres}^n(T)$. We claim now $M_d \in \mathrm{Add}\,T$ and then it holds that $M \in \widehat{\mathrm{Add}\,T}$.

In fact, note that $M_d \in \mathrm{Gen}(T)$ clearly, so we have an exact sequence

$$0 \to N \to T^{(H_T M_d)} \xrightarrow{\nu} M_d \to 0, \qquad (*)$$

where ν is the canonical evaluation map, which obviously remains exact after applying the functor H_T. By Lemma 2.3, $M_d \in \mathrm{Pres}^n(T) \subseteq T^{\perp}$. Hence we deduce that $N \in T^{\perp}$ too. Now by applying the functor $\mathrm{Hom}_R(-, N)$ to the exact sequence $0 \to M_d \to T_d \to \cdots \to T_1 \to M \to 0$, we obtain that

$$\mathrm{Ext}_R^1(M_d, N) \simeq \mathrm{Ext}_R^2(M_{d-1}, N) \simeq \cdots \simeq \mathrm{Ext}_R^d(M_1, N) \simeq \mathrm{Ext}_R^{d+1}(M, N),$$

by dimension shifting. Since $\mathrm{gd}\, R \leqslant d$, we see that $\mathrm{Ext}_R^1(M_d, N) \simeq \mathrm{Ext}_R^{d+1}(M, N) = 0$. It follows that the sequence $(*)$ splits and that $M_d \in \mathrm{Add}\,T$, as we claimed.

Since $\mathrm{Pres}^{n+d+1}(T) \subseteq \mathrm{Pres}^{n+d}(T) = \widehat{\mathrm{Add}\,T} \subseteq \mathrm{Pres}^{n+d+1}(T)$, we then get that

$$\mathrm{Pres}^{n+d}(T) = \mathrm{Pres}^{n+d+1}(T).$$

Note that T is clearly $(n+d+1)$-quasi-projective by Lemma 2.3 and that T is selfsmall by assumptions, so we have that T is a $*^{n+d}$-module [16]. □

Now we give the estimate of the global dimension of the endomorphism ring of a selforthogonal module.

Theorem 2.5. *Let R be a ring of finite global dimension and T be a selforthogonal R-module with $A = \mathrm{End}_R T$. Assume that* $\mathrm{id}_R T' \leqslant s$ *for all* $T' \in \mathrm{Add}\,T$. *Then* $\mathrm{fd}\, T_A \leqslant \mathrm{gd}\, A \leqslant s + \mathrm{fd}\, T_A$.

Proof. If $\mathrm{fd}\, T_A = \infty$, then $\mathrm{gd}\, A = \infty$ too and we have nothing to say in this case. So we assume that $\mathrm{fd}\, T_A = t < \infty$. Then it is obvious that we need only to show that $\mathrm{gd}\, A \leqslant s + t$.

For any A-module Y, by taking the projective resolution of Y, we obtain an exact sequence

$$0 \to Y_t \to P_{t-1} \to \cdots \to P_0 \to Y \to 0 \qquad (**)$$

with P_i projective for each $0 \leqslant i \leqslant t-1$. Denote by Y_i the ith syzygy, for each i. We claim now $\mathrm{pd}_A Y_t \leqslant s$ (note that $s < \infty$ since R is of finite global dimension) and so $\mathrm{pd}_A Y \leqslant t + s$. Then the conclusion will be followed from the arbitrarity of the choice of Y.

Indeed, by assumptions and Proposition 2.4, we have that $\mathrm{Pres}^m(T) = \widehat{\mathrm{Add}\,T}$ and T is a $*^m$-module for some m. Hence, by results in [16], we obtain that

$$\mathrm{Ker}\,\mathrm{Tor}_{i \geqslant 1}^A(T, -) = H_T\big(\mathrm{Pres}^m(T)\big).$$

Since $\mathrm{pd}_A H_T M \leqslant s$ for any $M \in \mathrm{Pres}^m(T)$ by Lemma 2.2, it follows that $\mathrm{pd}_A N \leqslant s$ for any $N \in \mathrm{Ker}\,\mathrm{Tor}^A_{i\geqslant 1}(T,-)$.

Now, by applying the functor $T \otimes_A -$ to the sequence $(**)$, we obtain that

$$\mathrm{Tor}^A_j(T, Y_t) \simeq \cdots \simeq \mathrm{Tor}^A_{j+t-1}(T, Y_1) \simeq \mathrm{Tor}^A_{j+t}(T, Y) \quad \text{for all } j \geqslant 1.$$

Since $\mathrm{fd}\, T_A \leqslant t < \infty$, we have that $\mathrm{Tor}^A_j(T, Y_t) \simeq \mathrm{Tor}^A_{j+t}(T, Y) = 0$ for all $j \geqslant 1$, i.e., $Y_t \in \mathrm{Ker}\,\mathrm{Tor}^A_{i\geqslant 1}(T,-)$. It follows that $\mathrm{pd}_A Y_t \leqslant s$ from arguments before, as desired. □

Restricting to some special cases, we have the following corollary.

Corollary 2.6. *Let R be a ring of finite global dimension and T an R-module with $A = \mathrm{End}_R T$. If moreover R is noetherian or T is of finitely generated projective resolution, then $\mathrm{fd}\, T_A \leqslant \mathrm{gd}\, A \leqslant \mathrm{id}_R T + \mathrm{fd}\, T_A$. In particular, if R is noetherian and T is injective, then $\mathrm{gd}\, A = \mathrm{fd}\, T_A = \mathrm{wgd}\, A$, where $\mathrm{wgd}\, A$ denotes the weak global dimension of A.*

Proof. If R is noetherian or T is of finitely generated projective resolution, then it always holds that $\mathrm{id}_R T' \leqslant \mathrm{id}_R T$ for all $T' \in \mathrm{Add}\, T$. Hence by applying Theorem 2.5, we have that $\mathrm{fd}\, T_A \leqslant \mathrm{gd}\, A \leqslant \mathrm{id}_R T + \mathrm{fd}\, T_A$.

If R is noetherian and T is injective, then clearly we have that $\mathrm{gd}\, A = \mathrm{fd}\, T_A$. By the definition of the weak global dimension, it always holds that $\mathrm{wgd}\, A \geqslant \mathrm{fd}\, T_A$. Since $\mathrm{wgd}\, A \leqslant \mathrm{gd}\, A$ obviously, it follows that $\mathrm{gd}\, A = \mathrm{fd}\, T_A = \mathrm{wgd}\, A$. □

3. Two questions on $*^n$-modules

In this section we will give answers to questions mentioned in the first section.

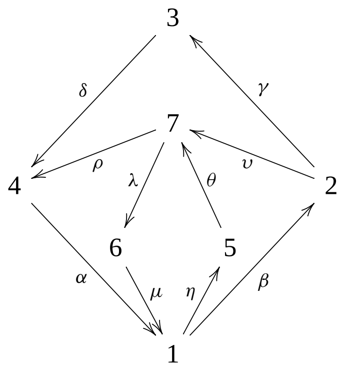
Firstly we note the following result.

Lemma 3.1. *Let R be a ring and T be a selforthogonal R-module with $\mathrm{pd}_R T \leqslant n$. If $\mathrm{fd}\, T_A = t < \infty$, where $A = \mathrm{End}_R T$, then T is a $*^m$-module for some m.*

Proof. By Lemma 2.3, we have that T is $(n+1)$-quasi-projective. If $\mathrm{fd}\, T_A = t = 0$, then T is a $*^2$-module by [15, Theorem 4.2]. So we assume that $t \geqslant 1$. Now by [15, Proposition 3.2], a selfsmall $(m, t+1)$-quasi-projective module T with the flat dimension of T over its endomorphism ring not more than t is a $*^m$-module. Note that T is obviously $(n+t+1, t+1)$-quasi-projective, so T is a $*^{n+t+1}$-module. □

Now we give an example of $*^3$-modules with infinite flat dimensions over their endomorphism rings. This shows that the answer to Question 1 in the first section is negative in general.

Example 3.2. Let R be the Artin algebra defined by the following quiver over a field k



with relations $\theta\lambda = 0 = \mu\eta$, $\gamma\delta = \upsilon\rho$, $\rho\alpha = \lambda\mu$, and $\beta\upsilon = \eta\theta$.

Then the R-module

$$T = \begin{array}{c} 3 \\ 2 \\ 1 \\ 4\ 6 \\ 3\ 7 \\ 2 \end{array} \oplus \begin{array}{c} 2 \\ 1 \\ 4\ 6 \\ 3\ 7 \\ 2 \end{array}$$

is a $*^3$-module with $\operatorname{fd} T_A = \infty$, where $A = \operatorname{End}_R T$.

Proof. Note that $\operatorname{gd} R = 2$ and T is indeed a projective R-module, so T is a $*^3$-module by Proposition 2.4.

If $\operatorname{fd} T_A = t < \infty$. Then $\operatorname{gd} A \leqslant 2 + t < \infty$ by Theorem 2.5. However, A is in fact the path algebra defined by the quiver

$$1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2$$

with the relation $\alpha\beta\alpha = 0$. It is easy to check that A is of infinite global dimension. Hence we see that $\operatorname{fd} T_A = \infty$. $\square$

More generally, we have the following result which also shows that the flat dimension of T_A for T a $*^n$-module ($n \geqslant 3$) with $A = \operatorname{End}_R T$ can even be arbitrarily far from the integer n.

Proposition 3.3. *Let A be an Artin algebra of finite representation type with* $\operatorname{gd} A = d$ (*maybe infinite*). *Then there exists an Artin algebra R with* $\operatorname{gd} R = 2$, *over which there is a $*^3$-module T with* $A = \operatorname{End}_R T$ *and* $\operatorname{fd} T_A = d$.

Proof. By a well-known result in the representation theory of Artin algebras, any Artin algebra of finite representation type can be obtained as the endomorphism algebra of a projective and injective module T over an Artin algebra R with $\operatorname{gd} R = 2$ (see for instance

[18]). In the case, we obtain that $A = \mathrm{End}_R T$ and that T is in fact a $*^3$-module by Proposition 2.4, since $\mathrm{gd}\, R = 2$ and $\mathrm{pd}_R T \leqslant 1$. Note that T is also injective, so we have that $\mathrm{fd}\, T_A = \mathrm{gd}\, A = d$ by Corollary 2.6. $\square$

However, in case $n = 2$, we have an affirmative answer to Question 1. To see this, we need the following lemma (cf. [5, Lemma 1.4]).

Lemma 3.4. *Let T be an R-module with $A = \mathrm{End}_R T$. Then*

$$\mathrm{Ker}\,\mathrm{Tor}_i^A(T, -) = \mathrm{Ker}\,\mathrm{Ext}_A^i\big(-, T^*\big) \quad \text{for each } i \geqslant 1.$$

In particular, $\mathrm{fd}\, T_A = \mathrm{id}_A T^*$.

Now we can show the following.

Proposition 3.5. *Let T be a $*^2$-module with $A = \mathrm{End}_R T$. Then T^* is 2-cotilting. In particular,* $\mathrm{fd}\, T_A \leqslant 2$.

Proof. Since T is a $*^2$-module, we have that $\mathrm{Pres}^2(T) \subseteq \mathrm{Stat}(T)$ by [16]. Since $\mathrm{Stat}(T) \subseteq \mathrm{Pres}^2(T)$ holds obviously, we get that $\mathrm{Stat}(P) = \mathrm{Pres}^2(T)$. Then by [17] or [12], we have that $\mathrm{Copres}\, T^* = \mathrm{Costat}(T) = H_T(\mathrm{Pres}^2(T))$.

By [16], it holds that

$$H_T\big(\mathrm{Pres}^2(T)\big) = \mathrm{Ker}\,\mathrm{Tor}_{i\geqslant 1}^A(T, -).$$

Hence we obtain that

$$\mathrm{Copres}\, T^* = \mathrm{Ker}\,\mathrm{Tor}_{i\geqslant 1}^A(T, -).$$

Now, Lemma 3.4 helps us to deduce that $\mathrm{Copres}\, T^* = \mathrm{Ker}\,\mathrm{Ext}_A^{i\geqslant 1}(-, T^*)$. It follows that T^* is 2-cotilting, by [2].

In particular, we have that $\mathrm{id}_A T^* \leqslant 2$ [2]. So, by Lemma 3.4, we also have that $\mathrm{fd}\, T_A \leqslant 2$, as we desired. $\square$

It is well known that, if T is a $*$-module, then T^* is 1-cotilting. Hence it is not surprise that T^* is 2-cotilting when T is a $*^2$-module. However, if T is a w-Σ-quasi-projective $*^2$-module, then T^* is also 1-cotilting, as the following result shows.

Corollary 3.6. *Let T be a $*^2$-module with $A = \mathrm{End}_R T$. If T is w-Σ-quasi-projective, then T^* is 1-cotilting. In particular,* $\mathrm{fd}\, T_A \leqslant 1$.

Proof. If T is w-Σ-quasi-projective, then we have that $\mathrm{Costat}(T) = H_T(\mathrm{Pres}^2(T)) = \mathrm{Cogen}\, T^*$ by [3]. Hence we obtain that

$$\mathrm{Ker}\,\mathrm{Tor}_{i\geqslant 1}^A(T, -) = H_T\big(\mathrm{Pres}^2(T)\big) = \mathrm{Cogen}\, T^*.$$

It follows that $\operatorname{Cogen} T^* = \operatorname{Ker} \operatorname{Ext}_A^{i\geqslant 1}(-, T^*)$ by Lemma 3.4. Then we have that T^* is 1-cotilting by [2]. In particular, we have that $\operatorname{fd} T_A = \operatorname{id}_A T^* \leqslant 1$. □

The following result classifies the flat dimension of T_A when T is a $*^2$-module with $A = \operatorname{End}_R T$.

Proposition 3.7. *Let T be a $*^2$-module with $A = \operatorname{End}_R T$. Then*

(1) $\operatorname{fd} T_A \leqslant 1$ *if and only if T is w-Σ-quasi-projective.*
(2) $\operatorname{fd} T_A = 0$ *if and only if T is semi-Σ-quasi-projective.*

Proof. (1) The sufficient part follows from Corollary 3.6. We now show the necessary part.

By the definition, we need to show that the functor H_T preserves the exactness of any exact sequence $0 \to M \to T_N \to N \to 0$ with $T_N \in \operatorname{Add} T$ and $M \in \operatorname{Gen}(T)$. After applying the functor H_T, we obtain two induced exact sequences $0 \to H_T M \to H_T T_N \to X \to 0$ and $0 \to X \to H_T N \to Y \to 0$ for some X, Y. Since T is a $*^2$-module, we have that $H_T T_N, H_T N \in \operatorname{Ker} \operatorname{Tor}^A_{i\geqslant 1}(T, -)$ by [16]. Hence, by dimension shifting, we obtain that

$$\operatorname{Tor}_1^A(T, X) \simeq \operatorname{Tor}_2^A(T, Y).$$

Since $\operatorname{fd} T_A \leqslant 1$, we see that $\operatorname{Tor}_1^A(T, X) = 0$ in fact. Hence we have the following commutative exact diagram, by applying the functor $T \otimes_A -$ to the exact sequence $0 \to H_T M \to H_T T_N \to X \to 0$:

$$\begin{tikzcd}
0 \arrow[r] & T \otimes_A H_T M \arrow[r] \arrow[d, "\rho_M"] & T \otimes_A H_T T_N \arrow[r] \arrow[d, "\rho_{T_N}"] & T \otimes_A X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r] & T_N \arrow[r] & N \arrow[r] & 0.
\end{tikzcd}$$

It follows that ρ_M is a monomorphism from the diagram. Since $M \in \operatorname{Gen}(T)$, ρ_M is also an epimorphism (see for instance [3]). Hence, ρ_M is an isomorphism. Therefore, we have that $M \in \operatorname{Stat}(P) \subseteq \operatorname{Pres}^2(T)$. Now since all three terms in the short exact sequence $0 \to M \to T_N \to N \to 0$ are in $\operatorname{Pres}^2(T)$, it remains exact after applying the functor H_T by [16], as we desired.

(2) By [15]. □

The following example shows that a w-Σ-quasi-projective $*^2$-module need neither be a $*$-module nor be semi-Σ-quasi-projective.

Example 3.8. Let R denote the path algebra defined by the quiver $1 \to 2 \to 3 \to 4$. Then, the partial tilting R-module

$$T = \begin{smallmatrix} 4 \\ 3 \\ 2 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 4 \\ 3 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 2 \end{smallmatrix}$$

is a w-Σ-quasi-projective $*^2$-module, which is neither a $*$-module nor semi-Σ-quasi-projective.

Proof. R is clearly a hereditary algebra, hence T is a w-Σ-quasi-projective $*^2$-module by Lemma 2.3 and Proposition 2.4, since T is partial tilting.

Since the R-module $3 \in \mathrm{Gen}(T)$ and $3 \notin \mathrm{Pres}^2(T)$, we have that $\mathrm{Gen}(P) \neq \mathrm{Pres}^2(T)$. Therefore, T is not a $*$-module.

Note also that we have an epimorphism

$$\begin{smallmatrix}4\\3\\2\\1\end{smallmatrix} \to \begin{smallmatrix}4\\3\\2\end{smallmatrix}$$

in $\mathrm{Add}\,T$, which cannot split, so T is not semi-Σ-quasi-projective by the definition. □

Combining results in Section 2, we have the following result.

Proposition 3.9. *Let R be a ring with $\mathrm{gd}\,R = 1$ and T be a selforthogonal R-module with $A = \mathrm{End}\,_R T$. Then*

(1) *T^* is 1-cotilting.*
(2) $\mathrm{gd}\,A \leqslant 2$.
(3) *If T is moreover semi-Σ-quasi-projective (specially T is projective) or injective, then* $\mathrm{gd}\,A \leqslant 1$.
(4) *If T is both semi-Σ-quasi-projective and injective, then A is a semisimple ring.*

Proof. By Lemma 2.3 and Proposition 2.4, we see that T is a w-Σ-quasi-projective $*^2$-module. Hence we have that T^* is 1-cotilting by Corollary 3.6. Note that $\mathrm{id}\,_R T' \leqslant 1$ for all $T' \in \mathrm{Add}\,T$ since $\mathrm{gd}\,R = 1$ by assumptions, so we obtain that $\mathrm{gd}\,A \leqslant 2$ by Theorem 2.5.

Similarly, we get that (3) and (4) hold by applying Proposition 3.7 and Theorem 2.5. □

Remark. In the representation theory of Artin algebras, it is well known that, for a partial tilting module T over a hereditary algebra R with $A = \mathrm{End}\,_R T$, the global dimension of the endomorphism algebra A is not more than 2. This is followed from the fact that A is indeed a tilted algebra, see for instance [8]. The last proposition generalizes this result. Moreover, it also gives a cotilting module over A which is obtained directly from the R-module T.

We end this paper with the following example, which shows that the answer to Question 2 in the first section is negative too in general, even in case $n = 2$.

Example 3.10. The abelian group $\mathbf{Q}$, as a $\mathbf{Z}$-module, is a $*^2$-module, which is clearly not finitely generated.

Proof. At first, $\mathbf{Q}$ is selfsmall since that, over a countable ring, any module with countable endomorphism ring is in fact selfsmall, by [1]. Secondly, $\mathbf{Q}$ is obviously a selforthogonal

Z-module since it is injective and **Z** is noetherian. Note also that $\mathrm{gd}\,\mathbf{Z} = 1$, so we have that **Q** is a $*^2$-module, by Proposition 2.4. □

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